Extending Constructive Solid Geometry to Projections and Parametric Objects

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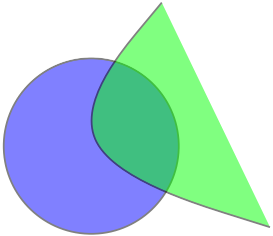
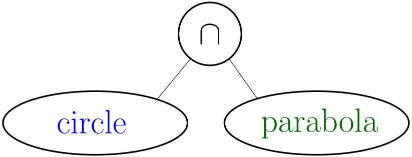
TMCE 2014
May 19–23, 2014, Budapest, Hungary

Constructive Solid Geometry

Boolean constructions via set operators:

- quantifier-free boolean formula
- atoms $P < 0$, $P = 0$, $P \in \mathcal{P}$
- operators $\neg, \vee, \wedge$ for complement A^c, $\cup$, $\cap$
- difference $A - B = A \cap B^c$, De Morgan...





CSG-tree representation

Motivation

Unified representation

- CSG primitives (boolean formulas)
- Projections (e.g. parametric solids)
- Minkowski Sums
- Extrusions or Sweeps

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Basic functionality

- Test if set is empty. Force a set to be non-empty
- Compute topology (simplicial complex homotopy equiv.)

Previous work (list non-exhaustive)

Topological properties

- semi-algebraic sets
  - Cylindrical Algebraic Decomposition [Collins, 1975]
    - semi-algebraic subsets homeomorphic to open boxes

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    - simplicial complex homotopy equivalent

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Topological properties

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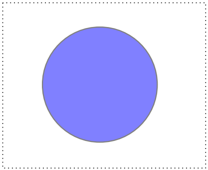
  - Cylindrical Algebraic Decomposition [Collins, 1975]
    - semi-algebraic subsets homeomorphic to open boxes
  - Interval Arithmetic [Delanoue et al, 2006]
    - simplicial complex homotopy equivalent
- implicit surfaces
  - Interval Analysis & Morse Theory [Hart et al, 1997]

Primitives

Assume $F(x, y) \leq 0$ (e.g. $x^2 + y^2 - 1 \leq 0$)

- $F(x, y) < 0 \rightarrow$ points "inside"
- $F(x, y) = 0 \rightarrow$ points on the boundary
- $F(x, y) > 0 \rightarrow$ points "outside"



$$x \quad : \quad [\ldots]$$
$$y \quad : \quad [\ldots]$$

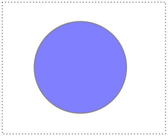
Primitives

"convert" inequality $F(x,y) \leq 0$ to equality by adding a slack variable:

$$F(x,y) - s = 0, \quad s \in (-\infty, 0]$$

$$F(x,y) \leq 0 \quad \longrightarrow \quad f(x,y;s) = 0$$

- $s < 0 \rightarrow$ points "inside"
- $s = 0 \rightarrow$ points on the boundary
- $s > 0 \rightarrow$ points "outside"
- s : characteristic variable of set A



current			new		
x	:	$[\ldots]$	s	:	$[a,b]$
y	:	$[\ldots]$			
$\vdots$					

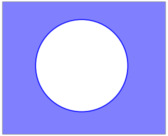
Complement

Given

$$A : \quad f(\mathbf{x}; s) = 0$$

We obtain

$$\neg A : \quad f(\mathbf{x}; -s) = 0$$



Remark

complement still represented by points from the manifold

Disjunctive Normal Form

Given primitives A, B and expressions P, Q_i

- A and $\neg A$ are in DNF
- $A \cup B$ and $A \cap B$ are in DNF
- $\neg(Q_1 \cup Q_2) = \neg Q_1 \cap \neg Q_2$
- $\neg(Q_1 \cap Q_2) = \neg Q_1 \cup \neg Q_2$
- $P \cap (Q_1 \cup Q_2 \cup \ldots \cup Q_n) = (P \cap Q_1) \cup (P \cap Q_2) \cup \ldots \cup (P \cap Q_n)$

Disjunctive Normal Form

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- $\neg(Q_1 \cup Q_2) = \neg Q_1 \cap \neg Q_2$
- $\neg(Q_1 \cap Q_2) = \neg Q_1 \cup \neg Q_2$
- $P \cap (Q_1 \cup Q_2 \cup \ldots \cup Q_n) = (P \cap Q_1) \cup (P \cap Q_2) \cup \ldots \cup (P \cap Q_n)$

projection distributes over union

$$\pi(Q_1 \cup Q_2 \cup \ldots \cup Q_n) = \pi(Q_1) \cup \pi(Q_2) \cup \ldots \cup \pi(Q_n)$$

Union

$$P = A_1 \cup \ldots \cup A_n$$

Consider each primitive separately:

$$\begin{aligned} A_1 &: f_1(\mathbf{x}; s_1) \\ A_2 &: f_2(\mathbf{x}; s_2) \\ &\vdots \\ A_n &: f_n(\mathbf{x}; s_n) \end{aligned}$$

Remark

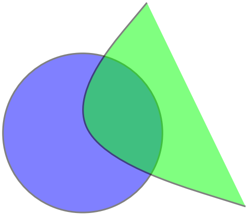
$\mathbf{x}$ may not be uniquely associated with a primitive

Intersection

$$P = A_1 \cap \ldots \cap A_n$$

Problem

x should be uniquely associated with a primitive

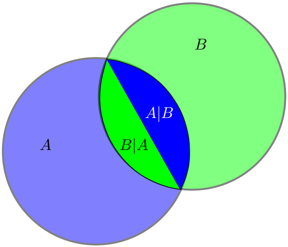


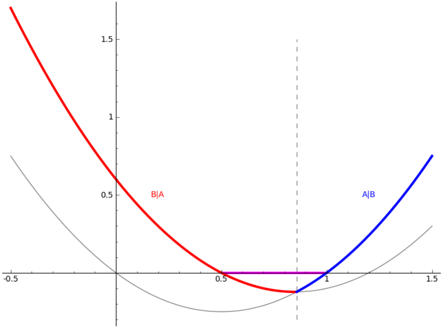
Intersection as union of disjoint sets (Dominant Set)

A dominates B: $A|B$

- $\mathbf{x} \in A \cap B$
- value of $A \geq$ value of B

$$A \cap B \; = \; A|B \; \cup \; B|A \qquad \text{(max)}$$

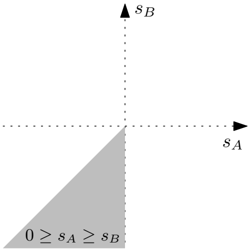




$$A|B_1, \ldots, B_n \; := \; \mathbf{x} \in \mathbb{R}^d : \; 0 \geq s_A \geq s_{B_i}$$

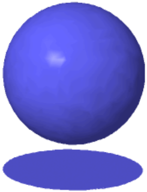
Properties of dominant sets

$$
\begin{aligned}
A \cap B &= A|B \;\cup\; B|A \\
(A|B)|C &= A|B, C \\
A|(B|C) \;\cup\; A|(C|B) &= A|B, C \\
A \cap B \cap C &= (A|B, C) \cup (B|C, A) \cup (C|A, B) \\
\neg(A|B) &= \neg A \;\cup\; \neg B \;\cup\; B|A
\end{aligned}
$$



Projection

$A : f(x, y, z; s) = 0$



$$\pi_z(A) = \begin{cases} \exists\, z : \\ f(x, y, z; s) = 0 \\ s \text{ is minimal} \end{cases}$$

Projection

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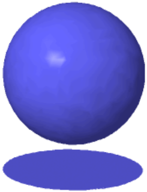

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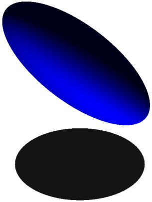
Lagrangian: $(f(x, y, z; s) = x^2 + y^2 + z^2 - 1 - s)$

$$L : \; s + \lambda f(x, y, z; s) \qquad \text{(works, but ugly!)}$$

$$(3\times3) : \begin{cases} L'_\lambda : & f(x, y, z; s) = 0 & & \\ L'_s : & 1 - \lambda = 0 & \Rightarrow & \lambda = 1 \\ L'_z : & \lambda \frac{\partial f(x,y,z;s)}{\partial z} = 0 & \Rightarrow & 2z = 0 \end{cases} \Rightarrow \begin{cases} s & = & x^2 + y^2 - 1 \\ \lambda & = & 1 \\ z & = & 0 \end{cases}$$

Projection

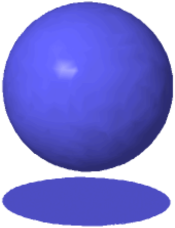


Theorem

Let $A : f(\mathbf{x}; s)$ be a geometric primitive. When projecting down k dimensions (eliminating $x_1 \ldots x_k$), the projection can be specified by:

$$\pi^k(A) \longrightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \ldots = \frac{\partial f}{\partial x_k} = 0$$

Projection (revisited)

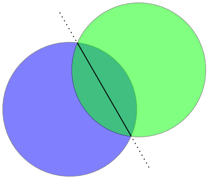


$$(2 \times 2) : \begin{cases} f(x,y,z;s) & = & 0 \\ \frac{\partial f(x,y,z;s)}{\partial z} & = & 0 \end{cases} \Rightarrow \begin{cases} s & = & x^2 + y^2 - 1 \\ z & = & 0 \end{cases}$$

Join Set

Primitives of more than one equation

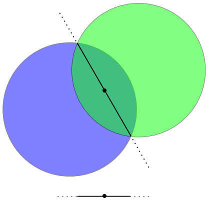
$$A \bowtie B \;:=\; \mathbf{x} \in \mathbb{R}^d : \; s_A = s_B \;\wedge\; s_A \leq 0$$



$$\begin{cases} f_A(\mathbf{x}; s_A) &= 0 \\ f_B(\mathbf{x}; s_B) &= 0 \\ s_A - s_B &= 0 \end{cases}$$

Join Set Projection

Recall that we minimize s wrt projected variables $x_i \dots$



- $\bowtie \to$ constraint (reduces dim.)
- even more constraints $\to$ Jacobian

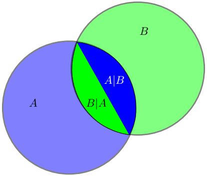
Lemma

Let $A : f_0(\mathbf{x}; s)$ and $B_i : f_i(\mathbf{x}; s), i = 1 \dots n$ geometric primitives. Then

$$\pi^k(A \bowtie B_1 \bowtie \cdots \bowtie B_n) \to \begin{cases} \emptyset, & k \leq n \\ J_{i_0 i_1 \dots i_n}(f_0, f_1, \dots, f_n) = 0, & k > n \end{cases},$$

where $1 \leq i_0 < i_1 < \dots < i_n \leq k$

Dominant Set Projection



Theorem

$$\pi^k(A|B) \;=\; \pi^k(A)|B \;\cup\; \pi^k(A \bowtie B)$$

Constrained optimization problem. Critical value of s:

- inside primitive
- boundary conditions (joins)

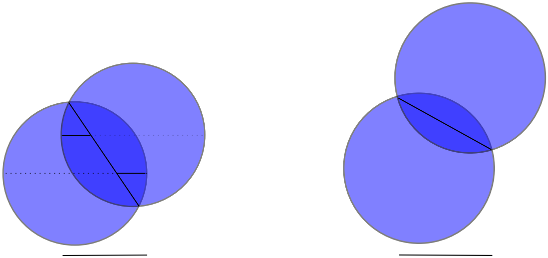
Dominant Set Projection (generalized)

Theorem

$\pi^k(A|[B_m]^{1:n}) =$

$$
\begin{array}{ll}
 & \pi^k(A)|[B_m]^{1:n} \\
\bigcup_{i=1}^{n} & \pi^k(A \bowtie B_i)|[B_m]^{1:n}_{m \neq i} \\
\bigcup_{\substack{i,j=1 \\ i<j}}^{n} & \pi^k(A \bowtie B_i \bowtie B_j)|[B_m]^{1:n}_{m \neq i, m \neq j} \\
\bigcup & \dots \\
\bigcup & \pi^k(A \bowtie B_1 \bowtie \cdots \bowtie B_n)
\end{array}
$$

Projection of Intersection



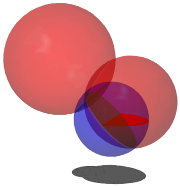
Basic transformation

$$\pi(A \cap B) \; = \; \pi(A)|B \; \cup \; \pi(B)|A \; \cup \; \pi(A \bowtie B)$$

- contributing points from each primitive (critical points of s)
- contributing points from joint (boundary condition)

Intersection of projection of intersections

$$\pi(E_1 \cap E_2) \ \cap \ \pi(E_1 \cap E_3)$$



$$\begin{aligned}
&= [\pi(E_1|E_2) \cup \pi(E_2|E_1)] \cap [\pi(E_1|E_3) \cup \pi(E_3|E_1)] \\
&= [\pi(E_1|E_2) \cap \pi(E_1|E_3)] \cup [\pi(E_1|E_2) \cap \pi(E_3|E_1)] \cup \\
&\quad [\pi(E_2|E_1) \cap \pi(E_1|E_3)] \cup [\pi(E_2|E_1) \cap \pi(E_3|E_1)] \\
&= [\pi(E_1)|E_2 \cup E_1 \bowtie E_2] \cap [\pi(E_1)|E_3 \cup E_1 \bowtie E_3] \cup \cdots \\
&= [\pi(E_1)|E_2 \cap \pi(E_1)|E_3] \cup \cdots \\
&= [\pi(E_1)|E_2, \pi(E_1)|E_3)] \cup [\pi(E_1)|E_3, \pi(E_1)|E_2] \cup \cdots \\
&= \bigcup_{i=1}^{18} S_i
\end{aligned}$$

$A \cap B = A|B \cup B|A$

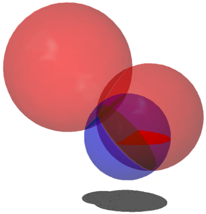
$(A \cup B) \cap (C \cup D) \rightarrow$ DNF

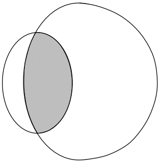
$\pi(A|B) = \pi(A)|B \cup \not\pi(A \bowtie B)$

Intersection of projection of intersections

$$\pi(E_1 \cap E_2) \ \cap \ \pi(E_1 \cap E_3)$$

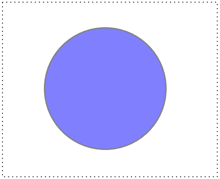
set	contributing set	formula
S_1	$\pi(E_1)$	$\pi(E_1)\|E_2, \pi(E_1)\|E_3$
S_2		$\pi(E_1)\|E_3, \pi(E_1)\|E_2$
S_3		$\pi(E_1)\|E_2, E_1 \bowtie E_3$
S_4		$\pi(E_1)\|E_3, E_1 \bowtie E_2$
S_5		$\pi(E_1)\|E_2, \pi(E_3)\|E_1$
S_6		$\pi(E_1)\|E_3, \pi(E_2)\|E_1$
S_7	$\pi(E_2)$	$\pi(E_2)\|E_1, \pi(E_1)\|E_3$
S_8		$\pi(E_2)\|E_1, E_1 \bowtie E_3$
S_9		$\pi(E_2)\|E_1, \pi(E_3)\|E_1$
S_{10}	$\pi(E_3)$	$\pi(E_3)\|E_1, \pi(E_1)\|E_2$
S_{11}		$\pi(E_3)\|E_1, E_1 \bowtie E_2$
S_{12}		$\pi(E_3)\|E_1, \pi(E_2)\|E_1$
S_{13}	$E_1 \bowtie E_2$	$E_1 \bowtie E_2\|(\pi(E_1)\|E_3)$
S_{14}		$E_1 \bowtie E_2\|E_1 \bowtie E_3$
S_{15}		$E_1 \bowtie E_2\|(\pi(E_3)\|E_1)$
S_{16}	$E_1 \bowtie E_3$	$E_1 \bowtie E_3\|(\pi(E_1)\|E_2)$
S_{17}		$E_1 \bowtie E_3\|E_1 \bowtie E_2$
S_{18}		$E_1 \bowtie E_3\|(\pi(E_2)\|E_1)$





Parametric Disk

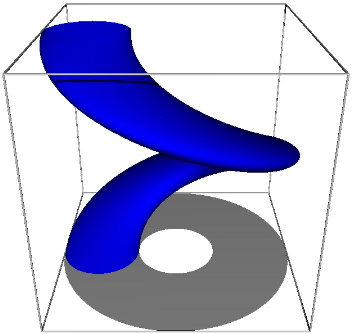
Via projections: project (eliminate) parameters



$$\begin{array}{rrcl} X: & x - r\ \cos\theta & = & 0 \\ Y: & y - r\ \sin\theta & = & 0 \\ R: & r^2 - 1 & = & s \end{array}, \quad \begin{array}{rcl} \theta & \in & [-\pi, \pi) \\ r & \in & [0, 1] \end{array}$$

$$\not\pi_{r,\theta}(R \bowtie X \bowtie Y)$$

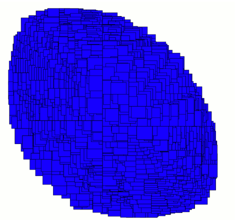
Parametric Annulus



$$F : (x - \cos t)^2 + (y - \sin t)^2 - \frac{1}{4} - s = 0 \quad \pi_t(F) \rightarrow \frac{\partial F}{\partial t} = 0$$

Python/SAGE

x,y,z = SR.var('x,y,z')
A=PrimitiveSet($(x-\frac{3}{4})^2+(y-\frac{3}{4})^2+(z-\frac{3}{4})^2 < \frac{2}{3}$,
{x:RIF(-2,2), y:RIF(-2,2),z:RIF(-2,2)})
B=PrimitiveSet($(x-\frac{1}{4})^2+(y-\frac{1}{4})^2+(z-\frac{1}{4})^2 < 1$,
{x:RIF(-2,2), y:RIF(-2,2),z:RIF(-2,2)})
G=ProjectionSet(IntersectionSet(A,B),set([z]))



Quimper [Chabert-Jaulin'09]

1.
$$\begin{cases} \frac{1}{16}(4z-3)^2+\frac{1}{16}(4y-3)^2+ \\ \quad +\frac{1}{16}(4x-3)^2-s_0-\frac{2}{3} = 0 \\ \frac{1}{16}(4z-1)^2+\frac{1}{16}(4y-1)^2+ \\ \quad +\frac{1}{16}(4x-1)^2-s_1-1 = 0 \\ 2z-\frac{3}{2}=0, \quad s_0-s_1 \geq 0 \end{cases}$$

2.
$$\begin{cases} \frac{1}{16}(4z-3)^2+\frac{1}{16}(4y-3)^2+ \\ \quad +\frac{1}{16}(4x-3)^2-s_0-\frac{2}{3} = 0 \\ \frac{1}{16}(4z-1)^2+\frac{1}{16}(4y-1)^2+ \\ \quad +\frac{1}{16}(4x-1)^2-s_1-1 = 0 \\ s_0-s_1 = 0 \end{cases}$$

3.
$$\begin{cases} \frac{1}{16}(4z-3)^2+\frac{1}{16}(4y-3)^2+ \\ \quad +\frac{1}{16}(4x-3)^2-s_0-\frac{2}{3} = 0 \\ \frac{1}{16}(4z-1)^2+\frac{1}{16}(4y-1)^2+ \\ \quad +\frac{1}{16}(4x-1)^2-s_1-1 = 0 \\ 2z-\frac{1}{2}=0, \quad s_1-s_0 \geq 0 \end{cases}$$

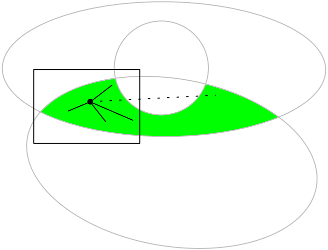
4. Identical to 2.

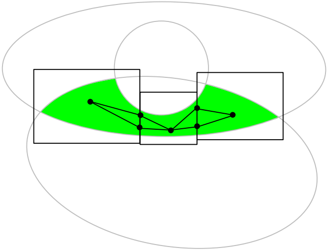
$$\pi(A\cap B)=\pi(A)|B\cup\pi(A\bowtie B)\cup\pi(B)|A\cup\pi(B\bowtie A)$$

Simplicial complex homotopy equivalent

[Delanoue et al, 2006] extension for more types of sets

- projections, Minkowski sums
- star test





Simplicial complex homotopy equivalent

[Delanoue et al, 2006] extension for more types of sets

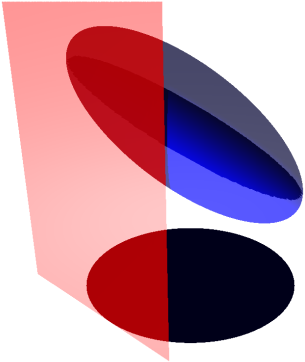
- projections, Minkowski sums
- star test

interior point s is a star

- if any segment from s lies inside
- if tangent planes at the boundary leave s on the same side (do not pass through!)

Work in progress/Future work

- Star test
  (boundary, gradient?)
- other constructions
  (medial axis)
- Minkowski Sums
  (characteristic variable?)



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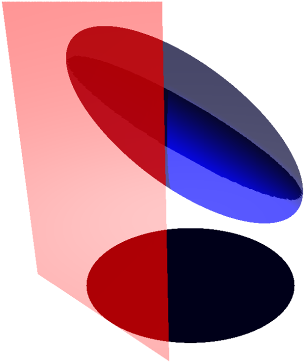

Thank you!